DETERMINATION OF TIME OF SOIL FREEZING BY METHOD OF CONCENTRIC ISOTHERMS

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A technique of determining the time of artificial freezing of soil in the presence of nonstationary temperature fields in the zones of freezing and cooling is discussed.

Two zones are formed in artificial freezing of soil around a hole:

1) an ice zone I with variable outer surface, on which the soil temperature is equal to 0°C;

2) a zone of cooled soil II, outside which the true temperature of the soil is observed within the accuracy of the measurements (±2°C). The temperature fields in these zones are formed practically by heat conduction.

Assuming zones I and II to be thermally isotropic, neglecting the effect of variation of layer pressure on the temperature variation, and taking the thermophysical parameters of the cooling agent and the soil to be constant, we have constructed the temperature fields [1] in these zones from the solution of the equations

$$\frac{\partial t_k(r,\ \tau)}{\partial \tau}=\frac{a_k}{r}\cdot\frac{\partial}{\partial r}\left(r\,\frac{\partial t_k(r,\ \tau)}{\partial r}\right),\ k=1,\ 2,\ \tau>0,\qquad (1)$$

$$R_1<r<\xi\ \text{ for }\ k=1,\ \xi<r<R_2\ \text{ for }\ k=2,$$

with the initial condition

$$t_k(r,\ 0)=t_0,\ k=1,\ 2\qquad (2)$$

and the boundary conditions

$$t_1(R_1\tau)=t_{\text{c}},\ t_k(\xi,\ \tau)=0,\ t_2(R_2,\ \tau)=t_0,\qquad (3)$$
$$\tau>0,\ \ k=1,\ 2.$$

If we introduce the nondimensional radius $m = r/R_1$, the problem (1)-(3) reduces to the solution of the equations

$$\frac{dt_k(m,\ \tau)}{d\tau}=\frac{a_k}{m}\cdot\frac{\partial}{\partial m}\left[m\,\frac{\partial t_k(m,\ \tau)}{\partial m}\right],\ k=1,\ 2,\ \tau>0,\qquad (4)$$

$$1<m<m_1\ \text{ for }\ k=1,\ m_1<m<m_1m_2\ \text{ for }\ k=2,$$

with the initial condition

$$t_k(m,\ 0)=t_0,\ k=1,\ 2\qquad (5)$$

and the boundary conditions

$$t_1(1,\ \tau)=t_{\text{C}},\ t_k(m_1,\ \tau)=0,\ t_2(m_1,\ m_2,\ \tau)=t_0,\qquad (6)$$
$$\tau>0,\ k=1,\ 2,$$

where

$$m_1=\frac{\xi}{R_1};\quad m_2=\frac{R_2}{\xi}.$$

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TABLE 1. Thermophysical Characteristics of Soils

Soil	W, %	σ·4.1868⁻¹, kJ/kg	$\lambda_1 \cdot 1.163^{-1}$, W/m·deg	$\lambda_2 \cdot 1.163^{-1}$, W/m·deg	ρ, kg/m³	a_1, m²/h	a_2, m²/h
Sand	23	80	2,7	2	10^3	0,0045	0,003
Clay	23	80	1,6	1,32	10^3	0,00234	0,00165
Alluvial rocks	22	80	2,34	1,7	10^3	0,00362	0,00357

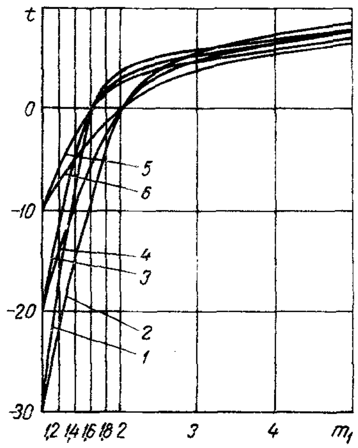

Fig. 1. Temperature distribution (°C) for alluvial sediments for $t_0 = 10°C$: 1) for $m_1 = 1.6$ and $t_c = -30°C$; 2) $m_1 = 2$ and $t_c = -30°C$; 3) $m_1 = 1.6$ and $t_c = -20°C$; 4) $m_1 = 2.0$ and $t_c = -20°C$; 5) $m_1 = 1.6$ and $t_c = -10°C$; 6) $m_1 = 2.0$ and $t_c = -10°C$.

For the solution of the problem (4)-(6) we use the method of solution of the problem (1)-(3) described in [1].

As a result the temperature fields in zones I and II for the case of nondimensional radius m are given by the formulas

$$t_1(m, \tau) = -\frac{t_c \ln \frac{m}{m_1}}{\ln m_1} + 2\sum_{n=1}^{\infty} \frac{1}{x_n D(x_n, m_1)} \{[J_0(mx_n) Y_0(m_1x_n) - J_0(m_1x_n) Y_0(mx_n)](t_0 - t_c) - t_0[J_0(mx_n) Y_0(x_n) - J_0(x_n) Y_0(mx_n)]\} \exp(-a_1 x_n^2 \tau), \quad (7)$$

$$t_2(m, \tau) = t_0 \frac{\ln \frac{m}{m_1}}{\ln m_2} - 2t_0 \sum_{n=1}^{\infty} \frac{1}{y_n D_1(y_n, m_2)} \left[J_0\left(\frac{m}{m_1} y_n\right) Y_0(m_2 y_n) - J_0(m_2 y_n) Y_0\left(\frac{m}{m_1} y_n\right)\right] \exp\left[-\left(\frac{y_n}{m_1}\right)^2 a_2 \tau\right], \quad (8)$$

where x_n and y_n are the roots of the characteristic equations

$$J_0(x) Y_0(m_1 x) - J_0(m_1 x) Y_0(x) = 0;$$
$$J_0(y) Y_0(m_2 y) - J_0(m_2 y) Y_0(y) = 0;$$
$$D(x_n, m_1) = m_1 C(x_n, m_1) - \Lambda(x_n, m_1);$$
$$D_1(y_n, m_2) = m_2 \Lambda(y_n, m_2) - C(y_n, m_2);$$
$$C(x_n, m_1) = J_1(m_1 x_n) Y_0(x_n) - J_0(x_n) Y_1(m_1 x_n);$$
$$\Lambda(x_n, m_1) = J_1(x_n) Y_0(m_1 x_n) - J_0(m_1 x_n) Y_1(x_n);$$

J_l and Y_l are Bessel functions of the first and second kind and of order l ($l = 0, 1$).

From formulas (7) and (8), and the data of Table 1, we determine the temperatures in zones I and II for sand, clay, and alluvial rocks.

The results of the computations are given in Table 2 and Fig. 1.

In the construction of the temperature fields in zones I and II for a hole of radius R_1 the quantity a_k ($k = 1, 2$) in formulas (7) and (8) must be replaced by a_k/R_1^2.

The problem of determining the freezing time of soil τ has been so far solved only for the case of a homogeneous stationary temperature field. We solve this problem for the case of nonstationary temperature fields (7) and (8) in zones I and II. Such an approach toward the determination of τ requires extremely complicated mathematical operations and laborious computations. Therefore we propose the method of concentric isotherms for the solution of this problem. The essence of the method is the following. Solving the equation

$$\left[\lambda_1 \frac{\partial t_1(m, \tau)}{\partial m} - \lambda_2 \frac{\partial t_2(m, \tau)}{\partial m}\right]_{m=m_1} = W\sigma\rho \frac{dm_1(\tau)}{d\tau}, \quad (9)$$

TABLE 2. Variation of Temperatures in Zones I and II for $t_0 = 20°C$ and $t_c = -20°C$

Soil	m	$m_1=1{,}6$	m	$m_1=2$
I II III	1,1	−15,8 −15,9 −15,9	1,1	−17,3 −17,3 −17,3
I II III	1,2	−12,2 −12,2 −12,2	1,2	−14,7 −14,7 −14,7
I II III	1,3	−8,8 −8,8 −8,8	1,3	−12,4 −12,4 −12,4
I II III	1,4	−5,67 −5,67 −5,67	1,4	−10,3 −10,3 −10,3
I II III	1,5	−2,4 −2,4 −2,4	1,5	−8,3 −8,3 −8,3
I II III	3,2	+10,98 +11,0 +10,45	1,6	−6,4 −6,4 −6,4
I II III	4,8	+14,75 +14,75 +14,55	1,7	−4,7 −4,7 −4,7
I II III	6,4	+15,13 +15,12 15,21	1,8	−3 −3 −3
I II III			1,9	−1,5 −1,5 −1,5
I II III			4	+12,58 +11,54 +11,98
I II III			6	+16,61 +15,97 +16,25
I II III			8	+19,71 +19,31 +19,51

Note: I) Sand; II) clay; III) alluvial sediments.

we obtain a formula for determining the freezing time of soil for

$$1+\frac{k}{10}<m_1\leqslant 1+\frac{k+1}{10},$$
$$k=0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9.$$

Differentiating formulas (7) and (8) with respect to m we find the temperature gradients in zones I and II for $m = m_1$:

$$\frac{\partial t_1(m_1\tau)}{\partial m}=-\frac{t_c}{m_1\ln m_1}+2\sum_{n=1}^{\infty}E_1(x_n,\ m_1,\ t_{c,0})\exp(-a_1x_n^2\tau),$$
$$\frac{\partial t_2(m_1\tau)}{\partial m}=\frac{t_0}{m_1\ln m_2}+\frac{2t_0}{m_1}\sum_{n=1}^{\infty}E_2(y_n,\ m_2)\exp\left[-\left(\frac{y_n}{m_1}\right)^2a_2\tau\right], \tag{10}$$

where

$$E_1(x_n,\ m_1,\ t_{c,0})=\frac{(t_c-t_0)B(x_n,\ m_1)+t_0C(x_n,\ m_1)}{D(x_n,\ m_1)};$$
$$E_2(y_n,\ m_2)=\frac{1}{m_2-\dfrac{C(y_n,\ m_2)}{\Lambda(y_n,\ m_2)}};$$
$$B(x_n,\ m_1)=J_1(m_1x_n)Y_0(m_1x_n)-J_0(m_1x_n)Y_1(m_1x_n).$$

Making use of equation (10), and using equation (9) we find the rate of freezing of the soil from the following:

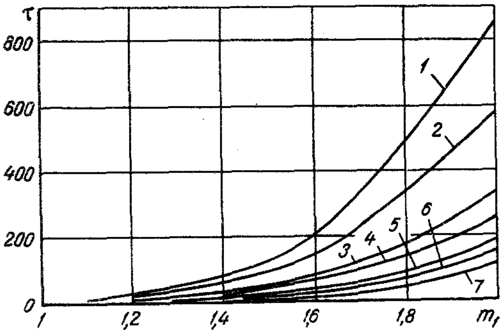

Fig. 2. Variation of freezing time (h) of alluvial sediments as a function of m_1 for different values of t_0 and t_C: 1) $t_C = -10°C$, $t_0 = 10°C$; 2) $t_C = -10°C$, $t_0 = 20°C$; 3) $t_C = -20°C$, $t_0 = 20°C$; 4) $t_C = -20°C$, $t_0 = 20°C$; 5) $t_C = -30°C$, $t_0 = 10°C$; 6) $t_C = -30°C$, $t_0 = 20°C$; 7) $t_C = -30°C$, $t_0 = 10°C$.

$$m_1\mu \frac{dm_1(\tau)}{d\tau} = -A(m_1, m_2) + 2m_1\lambda_1 \sum_{n=1}^{\infty} E_1(x_n, m_1, t_{c,0}) \exp(-a_1 x_n^2 \tau) - 2t_0\lambda_2 \sum_{n=1}^{\infty} E_2(y_n, m_2) \exp\left[-\left(\frac{y_n}{m_1}\right)^2 a_2\tau\right], \quad (11)$$

where

$$A(m_1, m_2) = \frac{t_c\lambda_1}{\ln m_1} + \frac{t_0\lambda_2}{\ln m_2}; \quad \mu = W\rho\sigma.$$

On the basis of experiments Kh. R. Khakimov [2] has assumed that $4.5 \leq m_2 \leq 5$. In view of this taking $m_2 = 5$ we rewrite Eq. (11) in the form

$$\frac{d\tau}{dm_1} = -\frac{\mu}{S(m_1, t_{c,0}, a_{1,2})}, \quad (12)$$

where

$$S(m_1, t_{c,0}, a_{1,2}) = \frac{1}{m_1}\Big\{A(m_1) - 2m_1\lambda_1 \sum_{n=1}^{\infty} E_1(x_n, m_1, t_{c,0}) \exp(-a_1 x_n^2 \tau) + 2t_0\lambda_2 \sum_{n=1}^{\infty} E_2 \exp\left[-\left(\frac{y_n}{m_1}\right)^2 a_2\tau\right]\Big\}; \quad (13)$$

$$A(m_1) = [A(m_1, m_2)]_{m_2=5};$$

$$E_2 = [E_2(y_n, m_2)]_{m_2=5}.$$

We solve Eq. (12) by the method of power series, i.e., in some neighborhood of each fixed value $m_1 = m_{1,0}$ we shall seek the solution of this equation in the form

$$\tau(m_1) = \tau_0 + \sum_{k=1}^{\infty} \frac{\tau^{(k)}(m_{1,0})}{k!} (\Delta m_1)^k, \quad (14)$$

where

$$\tau_0 = \tau(m_{1,0}); \quad \Delta m_1 = m_1 - m_{1,0}.$$

The investigated method of concentric isotherms depends on the rapid convergence of series (14).

Restricting series (14) to the first three terms we obtain the following formula to determine the freezing time of the soil for $m = m_1$:

$$\tau(m_1) = \tau_0 - z_0\Delta m_1 + \frac{z_0^2}{2\mu}(\Delta m_1^2)\Big\{A_1 - 2\lambda_1 \sum_{n=1}^{\infty}\Big[\Big(a_1 x_{n,0}^2 z_0 + \frac{2a_1 x_{n,0}^2 \tau_0}{m_{1,0} - 1} - \frac{m_{1,0} + 1}{(m_{1,0} - 1) D_0} x_{n,0} P_0 - \frac{1}{m_{1,0}}\Big) E_1^{(0)} + \frac{x_{n,0} t_0 P_0}{(m_{1,0} - 1) D_0}\Big] \exp(-a_1 x_{n,0}^2 \tau_0) + \frac{2t_0\lambda_2}{m_{1,0}^2} \sum_{n=1}^{\infty}\Big[\frac{a_2 y_n^2}{m_{1,0}^2}(2\tau_0 + m_{1,0} z_0) - 1\Big] E_2 \exp\Big[-a_2\Big(\frac{y_n}{m_{1,0}}\Big)^2 \tau_0\Big]\Big\}, \quad (15)$$

where

$$z_0 = \frac{\mu}{S_0}; \quad S_0 = S(m_{1,0}; t_{c,0}; a_{1,2}); \quad E_1^{(0)} = E_1(x_{n,0}; m_{1,0}; t_{c,0});$$

$$D_0 = D(x_{n,0}, m_{1,0}); \quad P_0 = P(x_{n,0}; m_{1,0});$$

$$A_1 = \frac{t_c\lambda_1(1 + \ln m_{1,0})}{(m_{1,0} \ln m_{1,0})^2} - \frac{t_0\lambda_2}{m_{1,0}^2 \ln 5};$$

$x_{n,0}$ are the roots of the equation

$$P(x_n, m_1) = J_1(m_1 x_n) Y_1(x_n) - J_1(x_n) Y_1(m_1 x_n);$$

$$J_0(x) Y_0(m_{1,0}; x) - J_0(m_{1,0}; x) Y_0(x) = 0.$$

TABLE 3. Freezing Time of Soil in Hours as a Function of m_1

m_1	$\tau_0 - z_0 \Delta m_1$	A	B	$\tau(m_1)$
1,1	0,049	0,527	$-2\cdot10^{-6}$	0,576
1,2	1,14	0,237	-10^{-4}	1,37
1,4	3,6	0,131	-10^{-3}	3,73
1,5	5,596	0,156	−0,002	5,75
1,6	8,682	0,226	−0,004	8,904
1,7	12,563	0,280	−0,006	12,837
1,8	17,297	0,3035	−0,0075	17,593
1,9	22,856	0,320	−0,009	23,187

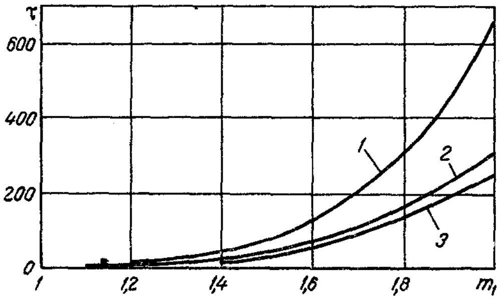

Fig. 3. Variation of freezing time (h) of different soils as a function of m_1 for $t_0 = 20°C$ and $t_c = -20°C$: 1) clay; 2) sand; 3) alluvial sediments.

The error ε due to discarding of the fourth and higher terms in series (14) is estimated as

$$\varepsilon = \left| \sum_{k=3}^{\infty} \frac{\tau^{(k)}(m_{1,0})}{k!} (\Delta m_1)^k \right|$$

$$\leqslant \sum_{k=3}^{\infty} |\tau^{(k)}(m_{1,0})| \left| \frac{(\Delta m_1)^k}{k!} \right| < M \sum_{k=3}^{\infty} \frac{1}{10^k k!}$$

$$< \frac{M}{4\cdot 10^3} \sum_{k=0}^{\infty} \frac{1}{40^k} = \frac{M}{39\cdot 10^2} < 0.00026M,$$

since $(\Delta m_1)^k < 10^{-k}$, and $k! > 4^k$ for $k \geq 3$; here

$$M \geqslant \left| \frac{d^k \tau(m_{1,0})}{dm_1^k} \right| = \left| \frac{d^k}{dm_1^k} \left(-\frac{\mu}{S(m_{1,0}, t_{c,0}, a_{1,2})} \right) \right|.$$

Hence by virtue of the relatively small rate of variation of the derivatives the error ε will be quite small and will have practically no effect on the results of computation of τ.

In practice formula (15) can be restricted to the first two terms of the series contained in it, since due to the presence of the exponential terms $\exp(-a_1 x_n^2 \tau)$ and $\exp[-(y_n/m_1)^2 a_2 \tau]$ all the terms of series (14) with the exception of the first two are close to zero.

For m = 1 the temperature t_c gets established suddenly at the initial instant. In view of this the rate of freezing of the soil at this instant will be infinitely large (this fact is also reflected by Eq. (9)). Therefore, in the determination of τ for $1.001 < m \leq 1.01$ we assume that in the first term of (15) $\tau(1.001) = 0$. In view of the smallness of R/1000 (for example, for $R_1 = 107$ mm this ratio is 0.107 mm) the error due to this assumption will not affect practical computations.

It follows from formula (15) that the increase of the freezing time of the soil

$$\Delta\tau_0 = \tau(m_1) - \tau(m_{1,0})$$

is directly proportional to $\mu = \rho\sigma W$. But σ and ρ are practically constant quantities: σ = 335 kJ/kg and ρ = 1000 kg/m³. Hence $\Delta\tau_0$ will be directly proportional to the moisture content of the soil W. Therefore it is sufficient to calculate τ from formula (15) for a given value of W, and for the remaining admissible values of W this result can be extended according to the rule of direct proportionality of the quantities $\Delta\tau_0$ and W.

As an example we calculated the freezing time of soil from the following data: $t_0 = +10°C$, $t_c = -40°C$, $\lambda_1 = 1.74$ W/m·deg, $\lambda_2 = 1.28$ W/m·deg, $a_1 = 10^{-3}$ m²/h, $a_2 = 10^{-3}$ m²/h, W = 10%, σ = 335 kJ/kg, ρ = 1000 kg/m³. The results of the computation are given in Table 3. Here

$$A = \frac{z_0^2}{2\mu} \Delta m_1^2 \Big\{ A_1 - 2\lambda_1 \sum_{n=1}^{\infty} \Big[\Big(a_1 x_{n,0}^2 z_0 + \frac{2a_1 x_{n,0}^2 \tau_0}{m_{1,0}-1} - \frac{m_{1,0}+1}{(m_{1,0}-1)D_0} x_{n,0} P_0 - \frac{1}{m_{1,0}} \Big) E_1^{(0)} + \frac{x_{n,0} t_0 P_0}{(m_{1,0}-1)D_0} \Big] \exp(-a_1 x_{n,0}^2 \tau_0) \Big\},$$

$$B = \frac{z_0^2 t_0 \lambda_2}{\mu m_{1,0}^2} \Delta m_1^2 \sum_{n=1}^{\infty} \left[\frac{a_2 y_n^2 (2\tau_0 + m_{1,0} z)}{m_{1,0}^2} - 1 \right] E_2 \exp\left[-a_2 \left(\frac{y_n}{m_{1,0}} \right)^2 \tau_0 \right].$$

Analyzing the results of computation of τ we arrive at the following conclusions:

a) if in formula (15) we discard the terms

$$\frac{z_0^2 t_0 \lambda_2}{m_{1,0}^2 \mu} (\Delta m_1)^2 \sum_{n=1}^{\infty} \left[\frac{a_2 y_n^2}{(m_{1,0})^2} (2\tau_0 + m_{1,0} z_0) - 1 \right] E_2 \exp\left[-a_2 \left(\frac{y_n}{m_{1,0}} \right)^2 \tau_0 \right],$$

then the error of the computation will be less than 0.04% of the general result. Therefore, in practical computations of the freezing time the following formula can be used:

$$\tau(m_1) = \tau_0 - z_0 \Delta m_1 + \frac{z_0^2}{2\mu} (\Delta m_1)^2 \Big\{ A_1 - 2\lambda_1 \sum_{n=1}^{\infty} \Big[\Big(a_1^2 z_0 x_{n,0}^2 + \frac{2a_1 x_{n,0}^2 \tau_0}{m_{1,0} - 1} - \frac{m_{1,0} + 1}{(m_{1,0} - 1) D_0} x_{n,0} P_0 - \frac{1}{m_{1,0}} \Big) E_1^{(0)} + \frac{x_{n,0} t_0 P_0}{(m_{1,0} - 1) D_0} \Big] \exp(-a_1 x_{n,0}^2 \tau_0) \Big\}; \quad (16)$$

b) for a less accurate determination of τ one can use the formula

$$\tau(m_{1,0}) = \tau_0 - z_0 \Delta m_1;$$

the error admitted here will be less than 5% of the result of computation from formula (15).

The results of computation of τ from formula (15) for soils indicated in Table 1 are presented in Figs. 2 and 3.

In the determination of the freezing time of soil around a hole of radius R_1 in formula (15) z_0 must be replaced by $z_0 R_1^2$ and a_k ($k = 1, 2$) by a_k/R_1^2.

From the results of the computation we conclude that:

1) the effect of the type of rock on the freezing time of soil is larger the greater the depth of freezing;

2) with the decrease of the temperature of the cooling agent the cooling time of the soil also decreases;

3) in the presence of variations of humidity of the soil the data of Fig. 3 can be used with a corresponding displacement of the time line, since it follows from formula (15) that with the increase of the moisture of the soil by a factor of k the growth of the freezing time of the soil also increases by a factor of k.

NOTATION

R_1	is the radius of the hole;
ξ	is the radius of the boundary between zones I and II;
R_2	is the outer radius of zone II;
t_0	is the true temperature of the soil;
t_c	is the temperature of the cooling agent;
a_1 and a_2	are coefficients of thermal diffusivity in zones I and II;
λ_1 and λ_2	are coefficients of thermal conductivity in zones I and II;
W	is the moisture content of the soil;
σ	is the latent heat of crystallization of water;
ρ	is the density of the soil.

LITERATURE CITED

1. V. I. Metenin and M. N. Shafeev, Heat and Mass Transfer [in Russian], Vol. II, Nauka i Tekhnika, Minsk (1968).
2. Kh. R. Khakimov, Freezing of Soils in Constructional Works [in Russian], Gosstroiizdat (1962).